

## ABSTRACT

### ANALYSIS OF MINIMUM-NORM SOLUTIONS IN NON-FULL RANK LINEAR REGRESSION MODELS USING THE MOORE–PENROSE INVERSE

By

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Linear regression models with a non-full rank design matrix lead to non-uniqueness of the ordinary least squares solution due to the singularity of  $X'X$ . Under such conditions, a mathematical approach is required to consistently select a single solution from the infinite set of least squares solutions. The Moore–Penrose inverse, as a generalized inverse, enables the derivation of a least squares solution even when the design matrix is non-full rank. This study aims to analyze and establish that the solution obtained through the Moore–Penrose inverse constitutes the minimum Euclidean norm element of the least squares solution set and is unique. The analysis employs a linear algebra framework involving column space, null space, orthogonal projection, and parameter estimability. To illustrate the results of the analysis, a numerical simulation was performed using SAS software on a singular design matrix formed by the relationship  $X_3 = X_1 + X_2$ , resulting in perfect multicollinearity. The results demonstrate that the Moore–Penrose solution corresponds to the orthogonal projection onto the column space and is orthogonal to the null space, thereby ensuring the minimum-norm and uniqueness properties. The simulations support the theoretical analysis and confirm the stability of the solution under perfect multicollinearity conditions.

**Keywords:** linear regression, non-full rank, Moore–Penrose inverse, minimum-norm solution, parameter estimability.

## ABSTRAK

### ANALISIS SOLUSI *MINIMUM-NORM* PADA MODEL REGRESI LINEAR *NON-FULL RANK* MENGGUNAKAN INVERS MOORE–PENROSE

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Model regresi linear dengan matriks desain *non-full rank* menyebabkan solusi metode *least squares* tidak bersifat unik akibat singularitas matriks  $X'X$ . Dalam kondisi tersebut, diperlukan pendekatan matematis yang mampu memilih satu solusi secara konsisten dari himpunan solusi yang tak hingga. Invers Moore–Penrose merupakan bentuk generalized inverse yang memungkinkan perolehan solusi *least squares* meskipun matriks desain *non-full rank*. Kajian ini bertujuan menganalisis serta membuktikan bahwa solusi yang dihasilkan melalui invers Moore–Penrose memiliki norma Euclidean minimum dari himpunan solusi *least squares* serta bersifat unik. Selanjutnya, analisis dilakukan melalui pendekatan aljabar linear yang mencakup kajian ruang kolom, ruang null, proyeksi ortogonal, dan estimabilitas parameter. Untuk mengilustrasikan hasil analisis, dilakukan simulasi numerik menggunakan perangkat lunak SAS pada matriks desain singular yang dibentuk dengan hubungan  $X_3 = X_1 + X_2$  sehingga terjadi multikolinearitas sempurna. Hasil analisis menunjukkan bahwa solusi Moore–Penrose merupakan proyeksi ortogonal terhadap ruang kolom matriks desain dan ortogonal terhadap ruang null, sehingga menjamin sifat minimum-norm dan keunikan solusi. Simulasi numerik memperkuat hasil teoretis dan menunjukkan kestabilan solusi pada kondisi multikolinearitas sempurna.

**Kata-kata kunci:** regresi linear, non-full rank, invers Moore–Penrose, minimum-norm, estimabilitas parameter.