Chapter 13

Dynamics of A Re-Parametrization of Two Dimensional Map $(x_{n+1}, y_{n+1}) = g_\mu(x_n, y_n) = \left(\frac{x_n(1-\mu x_n)}{y_n(x_n-\mu)}, x_n\right)$

Zakaria L.[1,a], Tuwankotta J.M.[2], Budhi M.W.S.[3]
[1]Lampung University, Soemantri Brojonegoro 1 Bandar Lampung, Indonesia
[2]Bandung Technology Institute, Ganeca 10 Bandung, Indonesia
[3]Bandung Technology Institute, Ganeca 10 Bandung, Indonesia
[a]zakaria@students.itb.ac.id

Abstract. The discussion of two dimensional mapping in this paper is based on a member of a family of system derived from a ΔΔ-sine Gordon equation introduced by J.M. Tuwankotta in 2005. By replacing the role of integrals and parameter in a system of difference equations, we will generate a new mapping and compare the properties of the new mapping with the original one, i.e. measure preserving property, their symmetries and reversing symmetry. Furthermore, the dynamics of the new mapping is analyzed.

Keywords: (Re-parametrization, two dimensional mapping, measure preserving property, possession of symmetries, reversing symmetries).

I. Introduction

In 1989, Quispel, Robert, and Thompson introduced the 12- parameter family of mapping of the plane given by [1, 6].

$$x_{n+2} = \frac{g_1(x_{n+1}) - x_n g_2(x_{n+1})}{g_2(x_{n+1}) - x_n g_3(x_{n+1})} \tag{1}$$

where g_j, $j = 1, 2$ is given by

$$\begin{pmatrix} g_1(x_{n+1}) \\ g_2(x_{n+1}) \\ g_3(x_{n+1}) \end{pmatrix} = A_0 \begin{pmatrix} x_{n+1}^2 \\ x_{n+1} \\ 1 \end{pmatrix} \times A_1 \begin{pmatrix} x_{n+1}^2 \\ x_{n+1} \\ 1 \end{pmatrix} \tag{2}$$

with A_0 and A_1 denoting arbitrary symmeric 3×3 matrices is given by

$$A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \beta_i & \epsilon_i & \sigma_i \\ \gamma_i & \sigma_i & \kappa_i \end{pmatrix}; i = 1, 2. \tag{3}$$

The symmeric QRT mappings (1) is an integrable reversible mappings of the plane. The mapping have (anti) measure preserving property. [4, 5, 6] Each member of the mapping (1) posseses an invariant or integral (that is : there exists a function G : R_2 → R such that G (xn, xn+1) = G (xn+1, xn+2) for all natural number n) is defined by a ratio of biquadratic polynomial of the form

$$G(x,y)=\frac{\alpha_0 x^2y^2+\beta_0\left(x^2y+xy^2\right)+\gamma_0\left(x^2+y^2\right)+\epsilon_0\left(x^2+y^2\right)+\sigma_0(x+y)+\kappa_0}{\alpha_1 x^2y^2+\beta_1\left(x^2y+xy^2\right)+\gamma_1\left(x^2+y^2\right)+\epsilon_1\left(x^2+y^2\right)+\sigma_1(x+y)+\kappa_1} \quad (4)$$

The following example is a special form of the QRT mapping. The mapping in this example will be focused in this article.

Example

Setting the symmetric matrices A_0 and A_1 as the following form

$$A_0=\begin{pmatrix} 0 & -1 & \mu \\ -1 & 1 & -1 \\ \mu & -1 & 0 \end{pmatrix};A_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5)$$

Hence, functions gi, i = 1, 2, 3 that corresponding to (2) can be written by

$$\begin{pmatrix} g_1\left(x_{n+1}\right) \\ g_2\left(x_{n+1}\right) \\ g_3\left(x_{n+1}\right) \end{pmatrix}=\begin{pmatrix} x_{n+1}^2-\mu x_{n+1}^3 \\ 0 \\ \mu x_{n+1}-x_{n+1}^2 \end{pmatrix} \quad (6)$$

By (5) and (6), we have the following mapping

$$x_{n+2}=\frac{x_{n+1}\left(1-\mu x_{n+1}\right)}{x_n\left(x_{n+1}-\mu\right)}, \quad \mu\in\mathbb{R} \quad (7)$$

For this example, notice that mapping (7) is integrable mapping that known as two-dimensional mapping derived from ΔΔ sine Gordon equation.[1, 2]. In article [2], we have studied a three parameters family of mappings which is derived from generalized sine-Gordon equation:

$$\theta_1\left(V_{l,m+1}V_{l+1,m}-V_{l+1,m+1}V_{l,m}\right)+\theta_2 V_{l+1,m+1}V_{l,m+1}V_{l+1,m}V_{l,m}=\theta_3. \quad (8)$$

A system of ordinary difference equations can be derived from (8) by restriction to traveling wave solution by setting

$$V_{l,m}=V_n,\ n=z_1 l+z_2 m,$$

where z_1 and z_2 are relatively prime integers. By substitution to (8) we derive

$$\theta_1\left(V_{n+z_2}V_{n+z_1}-V_{n+z_1+z_2}V_n\right)+\theta_2 V_{n+z_1+z_2}V_{n+z_2}V_{n+z_1}V_n=\theta_3 \quad (9)$$

which represents an infinite hierarchy of mapping labelled by z_1 and z_2. For fixed z_1 and z_2, equations (9) is mapping from $R^{z1+z2} \longrightarrow R^{z1+z2}$. Note that, the mapping in [1] can be obtained from (9) by setting θ_2 and θ_3 equal to 1 and θ_1 = pq.

II. Main Result

2.1. The Properties of Two Dimensional Mapping Derived from ΔΔ sine Gordon. Case Study: $\left(x_{n+1},y_{n+1}\right)=g_\mu\left(x_n,y_n\right)=\left(\frac{x_n(1-\mu x_n)}{y_n(x_n-\mu)},x_n\right)$

Let $z_1 = 1$, $z_2 = 2$, $\theta_1 = \mu\theta_2$, $\theta_3 = \theta_2$, and let us write

$$\gamma_n = \begin{pmatrix} V_{n+2}V_{n+1} \\ V_{n+1}V_n \end{pmatrix}$$

Of course, from (9), we have a two dimensional mapping below:

$$\gamma_{n+1} = \boldsymbol{g}_\mu(\gamma_n) \qquad (10)$$

Where

$$\begin{array}{llll} \boldsymbol{g}_\mu : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x, y) & \longmapsto & \left(\dfrac{x\,(1-\mu x)}{y\,(x-\mu)}, x\right) \end{array}$$

This mapping has an integral (that is : there exists a function $G : R^2 \rightarrow R$ such that $G(\gamma_{n+1}) = G(\gamma_n)$ for all natural number n). The explicit formula for the integral is

$$G_{(\mu,k)}(x, y) = \mu\left(\frac{x}{y} + \frac{y}{x} + 2k^2\right) - (x+y) - \left(\frac{1}{x} + \frac{1}{y}\right), \quad k \neq 0. \qquad (11)$$

Thus, for all n ϵ N, the solution of (10) lies on a level set of $G_{(\mu,k)}(x, y)$.

It is easy to show that (x, y) = (±1, ±1) and (x, y) = (±1, ∓1) points are fixed points and periodic-2 points of the mapping (10), respectively. The map (10) has the following properties: • The map (10) has an integral (11).

The map (10) is measure preserving:

$$|D\, g_{(\mu,1)}(x, y)| = \frac{x - x^2\mu}{y^2(x-\mu)} = \frac{\varrho(x, y)}{\varrho\left(\frac{x(1-\mu x)}{y(x-\mu)}, x\right)}$$

where ϱ is given by

$$\varrho(x, y) = \frac{1}{xy}$$

It is easy to show that

$$\varrho\left(\frac{x\,(1-\mu x)}{y\,(x-\mu)}, x\right) = \frac{y(x-\mu)}{x^2(1-x\mu)}$$

There exists a reversing symmetry L such that $L \circ g_\mu \circ L^{-1} = {}_{\mu}^{-1}$, namely

$$L(x, y) = (y, x)$$

In other words, the map (10) reversible because there exist L such that $g_\mu \circ L \circ g_\mu = L$. • There exists an involution S such that $S \circ g_\mu \circ S{-1} = -g_\mu$:

$$S(x, y) = (x, -y) .$$

We have a relation between mapping (10) together with an involution L and (7), namely

$$x_{n+2} = \frac{x_{n+1}(1-\mu x_{n+1})}{x_n(x_{n+1}-\mu)} \Leftrightarrow U_{SG}: x' = \frac{x(1-\mu x)}{y(x-\mu)}, y' = x; \quad L: x' = y, y' = x \tag{12}$$

It means two-dimensional mapping (10) that derived from ΔΔ−sine Gordon equations is a special case of 12-parameter family of QRT reversing symmeric integrable mapping. Note that twodimensional mapping (10) have singular lines (i.e. the line where the involutions are not defined),

and the symmetry line (i.e. the line fixed points of the involutions that make up the mapping), namely [5].

Singular lines : $yg_3(x) - g_2(x) = 0 \rightarrow y\,(x\mu + x^3) = 0.$

Symmetry lines : $y = x;\ y^2 g_3(x) - 2yg_2(x) + g_1(x) = 0 \rightarrow x = y;\ x\,(x - x^2\mu + y^2(-x + \mu)) = 0$

2.2. Re-parametrization of $(x_{n+1}, y_{n+1}) = g_\mu(x_n, y_n) = \left(\frac{x_n(1-\mu x_n)}{y_n(x_n-\mu)}, x_n\right)$

Consider the integral $G_{(\mu,k)}(x, y)$ for mapping (10). Note that $G_{(\mu,k)}(x, y)$ is linear in μ. Because of $G_{(\mu,k)}(x,y) = G_{(\mu,k)}\left(\frac{x(1-\mu x)}{y(x-\mu)}, x\right)$, then $G_{(\mu,k)}(x,y) = 0 \Longrightarrow G_{(\mu,k)}\left(\frac{x(1-\mu x)}{y(x-\mu)}, x\right)$. Therefore, for $k = \frac{1}{\sqrt{2|\mu|}}$, we have

$$\mu = \mu(x,y) = \frac{x+y-xy+x^2y+xy^2}{x^2+y^2} \tag{13}$$

And it follows that the map (10) with the replacement μ = μ(x, y) satisfies $\mu\left(\frac{x(1-\mu x)}{y(x-\mu)}, x\right) = \mu(x,y).$ Explicitly, the map (10) with the replacement μ = μ(x, y) yields the map,

$$\hat{g}_{(\mu)}(x,y) = -\frac{x\,(x + x^3 + x^2(-1+y) - y)}{x^3 + x(-1+y) - y - x^2 y}. \tag{14}$$

The map (14) has the following properties:

ˆ(μ, k) has an integral μ (x, y)

$$\mu(x,y) = \frac{x + y - xy + x^2y + xy^2}{x^2+y^2}. \tag{15}$$

ˆ(μ) measure-preserving, which means

where ϱ is given by $|D\,\hat{g}_{(\mu)}| = -\frac{\varrho(x,y)}{\varrho\left(\frac{x(1-\mu x)}{y(x-\mu)}, x\right)} = \frac{x^2\,(2 - 2x + x^2 - 2x^3 + 2x^4)}{(x^3 + x(-1+y) - y - x^2y)^2}$

$$\varrho(x,y) = \left[\partial_\mu G_{(\mu,k)}\right]^{-1} = \frac{1}{x^2+y^2}$$

It is easy to show that

$$\varrho\left(\frac{x(1-\mu x)}{y(x-\mu)},x\right)=\frac{\left(-x+x^3-y+xy-x^2y\right)^2}{x^2\left(2-2x+x^2-2x^3+2x^4\right)\left(x^2+y^2\right)}$$

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