COMPARISON OF SPATIAL AUTOREGRESSIVE (SAR) AND GEOGRAPHICALLY WEIGHTED REGRESSION (GWR) BASED ON SIMULATION STUDY

(Thesis)

By

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ABSTRACT

COMPARISON OF SPATIAL AUTOREGRESSIVE (SAR) AND GEOGRAPHICALLY WEIGHTED REGRESSION (GWR) BASED ON SIMULATION STUDY

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Spatial regression is an analysis that evaluates the relationship between one variable and several other variables that have spatial effects on several locations. There are two basic spatial concepts, namely spatial dependency and spatial heterogeneity. There are supervised learning techniques for regression that model spatial dependency, one of them is Spatial Autoregressive (SAR). In contrast to SAR, Geographically Weighted Regression (GWR) is a spatial regression method commonly used in data containing spatial heterogeneity. This study will compare which method is better between SAR and GWR for modeling spatial data if the data contains both spatial aspects, namely spatial dependency and spatial heterogeneity using simulation study. The simulation results of this study, based on bias, MSE and AIC of each model, it has been obtained that the SAR method is better than the GWR method for modeling data containing these two spatial aspects (spatial dependency and heterogeneity).

Keywords: Spatial, Dependency, Heterogeneity, SAR, GWR.

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CURRICULUM VITAE

The author's full name is Hilda Venelia who was born in Tanjung Karang on March 28, 1999. The author is the eldest of two children from Mr. Seven Aryadi and Mrs. Eliyawati.

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INSPIRATIONAL WORDS

"It may be that you dislike something, even though it is good for you, and it may be that you like something, even though it is not good for you. Allah knows while you don't know."

(Q.S Al-Baqarah: 216)

"Because the real hardship comes ease."

(Q.S Al-Insyirah: 5)

"Allah does not burden a person but according to his/her ability."

(Q.S Al-Baqarah: 286)

"The best of humans are those who are most beneficial to other human."

(HR. Ahmad, ath-Tharbani, ad-Daruqutni)

DEDICATION

Alhamdulillahirobbil'alamin,

Praise and gratitude I give to Allah Subhanahu Wata'ala because of the blessings and gifts,
Shalawat and greetings are always poured out to the Prophet Muhammad
Shallallahu 'Alaihi Wasallam that has given good news to mankind.

I dedicate this simple work to:

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There are no words that I can say to you except a big thank you for everything you have given me.
Love, affection, time, sacrifice, and sweat that I have not been able to repay.
Thank you for always praying for and supporting every step I choose.
Because the pleasure of Allah begins with your pleasure.

My Brother

Pray that I can become a better older sister.

My Friends

Thank you for all the support, happiness, jokes and laughter
what we've been through so far.

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Hopefully this thesis can provide many benefits for all of us. The writer also realizes that this thesis is still far from perfect, so the writer expects constructive criticism and suggestions to make this thesis even better.

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Author,

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I. INTRODUCTION

1.1 Background and Problem

Regression analysis is one of the statistical methods that can be used to find out the relationship between variables. Regression analysis depends on several assumptions that must be satisfied. The independence of error terms is a major assumption that is hardly met when data come from contiguous observations. When this assumption is violated, the inference on the coefficient becomes invalid in the conventional regression, i.e., Ordinary Least Square (OLS), due to inflated standard error. In addition, the heterogenous residual also allows the resulting model to be unable to explain the entire data. Problems like this usually arise when the data used is data that contains geographic elements, where the observation in one region can affect the observation in another adjacent region or the observation between adjacent regions has a correlation. Conditions that are influenced by spatial aspects or geographical conditions of a research area allow for spatial dependencies and spatial heterogeneity. So, with problems like this, it is no longer effective to use classical regression analysis. For this reason, one method that can be used on data containing spatial aspects is spatial regression analysis.

Spatial regression is an analysis that evaluates the relationship between one variable and several other variables that have spatial effects on several locations. Spatial regression is able to overcome the spatial aspect of the data by considering the contiguity of observations in the model. There are two basic spatial concepts, namely spatial dependency and spatial heterogeneity. Spatial dependency appears based on Tobler's first law that “everything is related to everything else, but near things are more related than distant things” and spatial heterogeneity is a spatial effect that shows the diversity between locations. There are supervised learning

techniques for regression that model spatial dependency, one of them is Spatial Autoregressive (SAR) [1]. SAR is one of spatial regression method with area approaches that combines a linear regression model with spatial lag on dependent variables using cross-section data and first introduced by Anselin [2]. Some studies have applied SAR models in various fields of science, such as [3–5]. For example, in the research of Hoef, et al. which uses the SAR model for inference statistics on ecological data [6]. Li and Zhou's research that aimed to look at factors affecting urban water quality in China using SAR models [7]. Koley and Bera's research that considered the testing for spatial dependence in SAR model with an endogenous regressor [8].

Another method of spatial regression analysis with a point approach is Geographically Weighted Regression (GWR) which was developed by Brunsdon, et al. [9]. In contrast to SAR, GWR is a spatial regression method commonly used in data containing spatial heterogeneity. Brunsdon, et al. also mentioned that GWR is a method by considering location elements as weights in estimating parameters so that each location has different regression parameters. Advantages GWR is the basis of the GWR framework which uses a classical regression framework and incorporates spatial relationships into the regression framework [10]. GWR analysis has also been widely used in various fields, such as in [11–15]. For example, Yacim and Boshoff's research that examined the influence of four spatial weighting functions and bandwidths on GWR using data on 3,232 house sales in Cape Town [16]. The research of Li, et al. that innovated geographically and temporally weighted co-location to the analysis of spatiotemporal crime patterns in greater Manchester using Monte Carlo simulation [17]. Then, the research of Wang, et al. that used GWR for simulating the spatial heterogeneity of housing prices in Wuhan, China [18].

In its application, the SAR method is able to model data containing spatial dependencies, while the GWR method is able to model data containing spatial heterogeneity. Both methods are good regression analysis methods for modeling spatial data with their respective problems. Therefore, in this study, the SAR and

GWR method will be compared for modeling spatial data when the data contains both spatial aspects, namely spatial dependency and spatial heterogeneity.

1.2 Research Objectives

The purpose of this research is to compare empirically the performance of SAR and GWR in spatial regression analysis when data containing spatial dependency and heterogeneity based on bias, Mean Square Error (MSE) and Akaike's Information Criterion (AIC).

1.3 Benefits of Research

The benefit of this study is to provide new references to readers about which model is better between SAR and GWR to overcome data containing spatial dependency and spatial heterogeneity based on the results of simulation study.

II. LITERATURE REVIEW

2.1 Linear Regression

Regression analysis is one of the statistical techniques commonly used to investigate the influence of independent variables (predictors) on dependent variables (responses). In general, the regression model for n observations with independent variable p can be written as follows [10]:

$$y_i = \beta_0 + \sum_{k=1}^{p} \beta_k x_{ik} + \varepsilon_i \tag{1}$$

with,

y_i : observation value of dependent variable on i-th observation,

x_{ik} : observation value of independent variable on i-th observation,

β_0 : intercept of regression model,

β_k : regression coefficient of k-th independent variable,

ε_i : error of i-th observation,

where $i = 1,2, \dots, n$ and assumed $\varepsilon_i \sim IIDN(0, \sigma^2)$, namely independent identically normal distributed of error.

Equation (1) it can be denoted in the form of the following matrices [19]:

$$\mathbf{y} = \mathbf{X\beta} + \boldsymbol{\varepsilon}$$

where,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1p} \\ 1 & x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \dots & x_{np} \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}, \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}.$$

The parameter estimation of β can be done using OLS method, by minimizing the sum of square of error so that obtained $\widehat{\boldsymbol{\beta}}$ that is an unbiased estimator for $\boldsymbol{\beta}$ [20]

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}}\mathbf{y} \tag{2}$$

which $\mathbf{X}$ is full column rank.

To find whether the model would be a good fit for the given data set, the coefficient of determination or R^2 is used and the formula is as follows:

$$R^2 = 1 - \frac{SSE}{SST} = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

where R^2 explains the proportion of the variance in the dependent variable that is predicted from the independent variables.

2.2 Spatial Regression

Spatial regression is an analysis that evaluates the relationship between one variable and several other variables that have spatial effects on several locations. The general model of spatial regression is stated in the following equation [2]:

$$\mathbf{y} = \rho\mathbf{W}_1\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

$$\mathbf{u} = \lambda\mathbf{W}_2\mathbf{u} + \boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$$

with,

$\mathbf{y}$: dependent variable matrix $(n \times 1)$,

$\mathbf{X}$: independent variable matrix $\big(n \times (k+1)\big)$,

$\boldsymbol{\beta}$: coefficient vector of parameter regression $\big((k+1) \times 1\big)$,

ρ : coefficient parameter lag spatial of dependent variable,

λ : coefficient parameter lag spatial in error$|\lambda| < 1$,

$\boldsymbol{u}, \boldsymbol{\varepsilon}$: error vector $(n \times 1)$,

$\boldsymbol{W}$: spatial weighting matrix $(n \times n)$,

where n is the number of observation or location and k is the number of independent variables.

2.3 Spatial Weighting Matrix

Spatial weighting matrix ($\mathbf{W}$) can be obtained based on distance information from the proximity of neighbor or in other words the distance between one region and another region. The common form of spatial weighting matrix ($\mathbf{W}$) is as follows [21]:

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}.$$

The elements of $\mathbf{W}$ above are w_{ij} with i is the row on element $\mathbf{W}$ and j is the column on element $\mathbf{W}$. Element $\mathbf{W}$ above consists of two values, namely zero or one, where the value $w_{ij} = 1$ for the region adjacent to the location of observation and $w_{ij} = 0$ for the region not adjacent to the location of observation.

There are several methods to determine the spatial weighting matrix. One of them is the Queen Contiguity method. The calculation of the weighting matrix using the Queen Contiguity method is illustrated in the following figure [22].

	1	2	3	
	4	5	6	
	7	8	9	

Figure 1. Queen Contiguity method illustration.

In Figure 1, nine regions are illustrated as observations. An element of the matrix is defined as 1 if the common side and common vertex are adjacent to the area of interest. The other regions are defined as elements of the matrix with a value of 0. For example, for region number 5 obtained $w_{51} = w_{52} = w_{53} = w_{54} = w_{56} = w_{57} = w_{58} = w_{59} = 1$ and the others are 0. Whereas, for region number 9 obtained $w_{95} = w_{96} = w_{98} = 1$ and the others are 0. Then, $\mathbf{W}$ order 9×9 has the following form.

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

2.4 Spatial Data Aspects

Modeling on spatial data can be grouped based on the type of spatial data used, namely spatial point and spatial area. In analyzing spatial data, there are two basic spatial concepts, namely spatial dependency that looks at the dependence between observations and spatial heterogeneity that looks at diversity between observations. Spatial analysis is done if the data used meets the spatial aspect, which has a correlated error or has spatial heterogeneity.

2.4.1 Spatial Dependency

Spatial dependency tests are conducted to see if observations at one location affect observations in other adjacent locations. Spatial dependency testing is performed with Moran's I test with the following hypotheses [22].

$H_0 : I = 0$ (there is no spatial dependency)

$H_1 : I \neq 0$ (there is the spatial dependency)

The statistical test of Moran's I index are derived in the random variable statistics of standard normal. It is based on Central Limit Theorem whereby for the large n and variances is known, then Z_I will distributed with standard normal as follows:

$$Z_I = \frac{I - E(I)}{\sqrt{Var(I)}} \tag{3}$$

with,

I : Moran's I index,

Z_I : the statistical test value of Moran's I index,

$E(I)$: the expected value of Moran's I index,

$Var(I)$: the variances of Moran's I index.

$$I = \frac{\hat{\boldsymbol{\varepsilon}}^{\mathrm{T}}\mathbf{W}\hat{\boldsymbol{\varepsilon}}}{\hat{\boldsymbol{\varepsilon}}^{\mathrm{T}}\hat{\boldsymbol{\varepsilon}}}$$

$$I_0 = E(I) = \frac{tr(\mathbf{MW})}{n-p}$$

$$Var(I) = \frac{tr(\mathbf{MWMW}^{\mathrm{T}}) + tr(\mathbf{MWMW}) - \left(tr(\mathbf{MW})\right)^2}{(n-p)(n-p-2)} - [E(I)]^2$$

where,

$\mathbf{M} = [\mathbf{I} + \mathbf{X}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}]$ and the value of index I in between -1 and 1.

The area of rejection is rejected H_0 if $|Z_I| > Z_{\alpha/2}$ or $p - value < \alpha$. If $\hat{I} > \hat{I}_0$ means the data has a positive autocorrelation and if $\hat{I} < \hat{I}_0$ means the data has a negative autocorrelation [23].

2.4.2 Spatial Heterogeneity

Spatial heterogeneity occurs due to differences in characteristics of one region with another region. The location position of an observation allows for a relationship with other observations that are close to each other. Spatial heterogeneity testing is conducted with Breusch-Pagan test statistics with the following hypothesis [2].

$H_0 : \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2 = \sigma^2$

H_1 : at least one $\sigma_i^2 \neq \sigma^2$

Statistical test:

$$BP = \left(\frac{1}{2}\right)\boldsymbol{f}^T\boldsymbol{Z}(\boldsymbol{Z}^T\boldsymbol{Z})^{-1}\boldsymbol{Z}^T\boldsymbol{f} \tag{4}$$

where $\boldsymbol{f} = (f_1, f_2, \dots, f_n)^T$ with $f_i = \left(\frac{e_i^2}{\sigma^2} - 1\right)$ and $e_i = y_i - \hat{y}_i$ is least square residual for i-th observation. $\boldsymbol{Z}$ is a matrix $\left(n \times (p+1)\right)$ that contains vectors that have been standardized for each observation.

The area of rejection is rejected H_0 if $BP > \chi_p^2$ or if $p - value < \alpha$ with p is the number of independent variables.

2.5 Spatial Autoregressive (SAR)

Spatial Autoregressive Model (SAR) or Spatial Lag Model (SLM) is a model that combines linear regression models with spatial lag on dependent variables using cross-section data [2]. Spatial lag arises when the observation value of a dependent variable at a location correlates with the observation value of the dependent variable in the surrounding location or in other words there is a spatial correlation between dependent variables. In this model there is a function of the dependent variable at j location which is used as an independent variable to predict the value of the dependent variable at i location. SAR model is generally written as follows [24].

$$\mathbf{y} = \rho\mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \tag{5}$$

$$\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$$

2.5.1 Parameter Estimation of SAR Model

Parameter estimation in SAR model can be done through Maximum Likelihood Estimation (MLE) method. The first step is to form the likelihood function of Equation (5). The formation of likelihood function is done with ε so that is obtained as follows:

$$\mathbf{y} = \rho\mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} = \mathbf{y} - \rho\mathbf{W}\mathbf{y} - \mathbf{X}\boldsymbol{\beta}$$

$$\boldsymbol{\varepsilon} = (\mathbf{I} - \rho\mathbf{W})\mathbf{y} - \mathbf{X}\boldsymbol{\beta} \tag{6}$$

$$L(\sigma^2; \varepsilon) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon})\right) \tag{7}$$

$$L(\rho, \boldsymbol{\beta}, \sigma^2|\boldsymbol{y}) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} (J) \exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon})\right) \tag{8}$$

with $J = \left|\frac{\partial\boldsymbol{\varepsilon}}{\partial\boldsymbol{y}}\right| = |\mathbf{I} - \rho\mathbf{W}|$ is Jacobian function, which derivative of Equation (6) of $\boldsymbol{y}$. Then substitute Equation (6) to Equation (8) so that is obtained:

$$L(\rho, \boldsymbol{\beta}, \sigma^2|\boldsymbol{y}) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} |\mathbf{I} - \rho\mathbf{W}| \ \exp\left(-\frac{1}{2\sigma^2}\left((\mathbf{I} - \rho\mathbf{W})\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\right)^T\left((\mathbf{I} - \rho\mathbf{W})\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\right)\right)$$

$$\ln(L) = \frac{n}{2}\ln\left(\frac{1}{2\pi\sigma^2}\right) + \ln|\mathbf{I} - \rho\mathbf{W}| - \frac{1}{2\sigma^2}\left((\mathbf{I} - \rho\mathbf{W})\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\right)^T\left((\mathbf{I} - \rho\mathbf{W})\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\right)$$

$$\ln(L) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) + \ln|\mathbf{I} - \rho\mathbf{W}| - \frac{1}{2\sigma^2}\left(\left((\mathbf{I} - \rho\mathbf{W})\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\right)^T\left((\mathbf{I} - \rho\mathbf{W})\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\right)\right).$$

Furthermore, the estimation of parameter $\boldsymbol{\beta}$ obtained by maximizing the natural logarithm function, which derivative the equation of $\boldsymbol{\beta}$ so that can be expressed as follows.

$$\frac{d}{d\boldsymbol{\beta}}\ln(L) = -2\mathbf{X}^T(\mathbf{I} - \rho\mathbf{W}_1)\mathrm{y} + 2\mathbf{X}^T\mathbf{X}\widehat{\boldsymbol{\beta}} = \mathbf{X}^T(\mathbf{I} - \rho\mathbf{W}_1)\mathrm{y} - \mathbf{X}^T\mathbf{X}\widehat{\boldsymbol{\beta}} = 0$$

Therefore, the estimator of parameter SAR model is obtained

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}(\mathbf{I} - \rho\mathbf{W})\mathbf{y} \tag{9}$$

or $\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}}_G - \rho\widehat{\boldsymbol{\beta}}_L$ with $\widehat{\boldsymbol{\beta}}_G = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y}$ and $\widehat{\boldsymbol{\beta}}_L = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{y}$. Such that, we also find $\hat{\boldsymbol{\varepsilon}}_G = \mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_G$ and $\hat{\boldsymbol{\varepsilon}}_L = \mathbf{W}_1\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_G$.

While the natural logarithm function to estimate ρ is

$$\ln\big(L(\rho)\big) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\left(\frac{[e_0 - \rho e_d]^T[e_0 - \rho e_d]}{n}\right) + \ln|\mathbf{I} - \rho\mathbf{W}| - \frac{1}{2}$$

$$\ln\big(L(\rho)\big) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\{[e_0 - \rho e_d]^T[e_0 - \rho e_d]\} - \frac{n}{2}\ln(n) + \ln|\mathbf{I} - \rho\mathbf{W}| - \frac{1}{2}$$

so, the parameter estimate $\hat{\rho}$ obtained by optimizing the following equation

$$f(\rho) = C - \frac{n}{2}\ln\{[e_0 - \rho e_d]^T[e_0 - \rho e_d]\} + \ln|\mathbf{I} - \rho\mathbf{W}|$$

where,

$$C = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(n) - \frac{1}{2},$$

$e_0 = \hat{\boldsymbol{\varepsilon}}_G$ dan $e_d = \hat{\boldsymbol{\varepsilon}}_L$,

$$\hat{\sigma}^2 = \frac{[e_0 - \rho e_d]^T[e_0 - \rho e_d]}{n}.$$

2.6 Geographically Weighted Regression (GWR)

Geographically Weighted Regression (GWR) is a model developed by Brunsdon, et al. [9] for continuous dependent variables. This model is a local regression model that produces a local model parameter estimator for each point or location where the data is collected such that each location point has different regression parameters. In GWR model, dependent variable is predicted by independent

variable whose regression coefficient depends on the location where the data is observed. GWR model can be written as follows [10]:

$$y_i = \beta_0(u_i, v_i) + \sum_{k=1}^{p} \beta_k(u_i, v_i) x_{ik} + \varepsilon_i \tag{10}$$

where,

y_i : observation value of dependent variable for i-th location,

x_{ik} : observation value of k-th independent variable for i-th location,

$\beta_0(u_i, v_i)$: intercept of GWR model,

$\beta_k(u_i, v_i)$: regression coefficient of k-th independent variable for i-th location,

(u_i, v_i) : coordinate of geographical location namely longitude (long) and latitude (lat) for i-th location,

ε_i : error of i-th observation and follow an independent normal distribution with zero mean and variance σ^2 [25].

2.6.1 Parameter Estimation of GWR Model

In the estimation of parameters in the GWR model is used the Weighted Least Square (WLS) method, which is to provide different weighting at each location where the data is observed. Suppose the weighting for location (u_i, v_i) is $w_j(u_i, v_i)$ $(j = 1,2, \dots, n)$, then parameter for observation of location (u_i, v_i) is estimated by adding weighting elements $w_j(u_i, v_i)$ in Equation (10). Then minimize the sum squares of error so that the following equation is obtained.

$$\sum_{j=1}^{n} w_j(u_i, v_i)\varepsilon_j^2 = \sum_{j=1}^{n} w_j(u_i, v_i)\left[y_j - \beta_0(u_i, v_i) - \sum_{k=1}^{p} \beta_k(u_i, v_i) x_{ik}\right]^2$$

Therefore, the estimation of parameters with the WLS method is obtained from the equation in the following matrix.

$$\boldsymbol{\varepsilon}^T \mathbf{W}(u_i, v_i)\boldsymbol{\varepsilon} = [\mathbf{y} - \mathbf{X}\boldsymbol{\beta}(u_i, v_i)]^T \mathbf{W}(u_i, v_i)[\mathbf{y} - \mathbf{X}\boldsymbol{\beta}(u_i, v_i)]$$

$$\boldsymbol{\varepsilon}^T \mathbf{W}(u_i, v_i)\boldsymbol{\varepsilon} = \mathbf{y}^T \mathbf{W}(u_i, v_i)\mathbf{y} - \mathbf{y}^T \mathbf{W}(u_i, v_i)\mathbf{X}\boldsymbol{\beta}(u_i, v_i) - \boldsymbol{\beta}^T(u_i, v_i)\mathbf{X}^T \mathbf{W}(u_i, v_i)\mathbf{y} + \boldsymbol{\beta}^T(u_i, v_i)\mathbf{X}^T \mathbf{W}(u_i, v_i)\mathbf{X}\boldsymbol{\beta}(u_i, v_i)$$

$$\boldsymbol{\varepsilon}^T \mathbf{W}(u_i, v_i)\boldsymbol{\varepsilon} = \mathbf{y}^T \mathbf{W}(u_i, v_i)\mathbf{y} - 2(\boldsymbol{\beta}^T(u_i, v_i)\mathbf{X}^T \mathbf{W}(u_i, v_i)\mathbf{y}) + \boldsymbol{\beta}^T(u_i, v_i)\mathbf{X}^T \mathbf{W}(u_i, v_i)\mathbf{X}\boldsymbol{\beta}(u_i, v_i)$$

$$\boldsymbol{\varepsilon}^T \mathbf{W}(u_i, v_i)\boldsymbol{\varepsilon} = \mathbf{y}^T \mathbf{W}(u_i, v_i)\mathbf{y} - 2\boldsymbol{\beta}^T(u_i, v_i)\mathbf{X}^T \mathbf{W}(u_i, v_i)\mathbf{y} + \boldsymbol{\beta}^T(u_i, v_i)\mathbf{X}^T \mathbf{W}(u_i, v_i)\mathbf{X}\boldsymbol{\beta}(u_i, v_i)$$

Then it is derived by $\boldsymbol{\beta}^T(u_i, v_i)$ and obtained as follows:

$$-2\mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{y} + 2\mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{X}\boldsymbol{\beta}(u_i, v_i) = 0$$

$$-2\mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{y} = -2\mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{X}\boldsymbol{\beta}(u_i, v_i)$$

$$\mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{X}\boldsymbol{\beta}(u_i, v_i) = \mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{y}$$

$$\boldsymbol{\beta}(u_i, v_i) = (\mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{y}.$$

So that obtained the estimator of parameter in GWR model for each location is [9]

$$\widehat{\boldsymbol{\beta}}(u_i, v_i) = (\mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}(u_i, v_i)\mathbf{y}, \tag{11}$$

if there are n sample location then this estimation is the estimation of each row from parameter local matrix of the entire location and the matrix is as follows:

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0(u_1, v_1) & \beta_1(u_1, v_1) & \beta_2(u_1, v_1) & \dots & \beta_p(u_1, v_1) \\ \beta_0(u_2, v_2) & \beta_1(u_2, v_2) & \beta_2(u_2, v_2) & \dots & \beta_p(u_2, v_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_0(u_n, v_n) & \beta_1(u_n, v_n) & \beta_2(u_n, v_n) & \dots & \beta_p(u_n, v_n) \end{bmatrix}.$$

The weighting matrix is diagonal matrix that shows a varied weighting of each parameter estimate in i-th location that formulated as follows:

$$\mathbf{W}(u_i, v_i) = \begin{bmatrix} w_{i1} & 0 & \dots & 0 \\ 0 & w_{i2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_{in} \end{bmatrix}.$$

2.6.2 Weighting of GWR Model

In spatial analysis, the estimation of parameters at a point (u_i, v_i) will be more influenced by points close to the location (u_i, v_i) than by more distant points. Spatial weighting selection is used to determine the weighting size of each different location. The role of spatial weighting is very important because the weighting value represents the location of observation data with each other. The location that close to the observed location are given large weighting while distant ones are given small weighting [23]. There are several functions that can be used to determine the weighting size for each different location on the GWR model, including inverse distance and kernel function.

Kernel function is used to estimate the parameter of GWR model if the distance function (w_j) is continuous and monotonically decreasing [26]. Kernel functions

that can be used to form the weights are Gaussian, Bisquare and Tricube functions. Gaussian function can be written as follows:

$$w_j(u_i, v_i) = \exp\left(-\frac{1}{2}\left(\frac{d_{ij}}{b}\right)^2\right)$$

where d_{ij} is the distance between location (u_i, v_i) and (u_j, v_j) and b is bandwidth. Bandwidth is the radius of a circle where points within that radius are still considered influential in forming model parameter of i-th location. Very small bandwidth will cause very large variance. That is because if the bandwidth is very small then there will be fewer observations that are within radius b so that the model obtained is under smoothing because the estimation results use few observations. Conversely, if the bandwidth gets bigger then it gives rise to a greater bias. If the bandwidth is very large then more and more observations are within radius b so that the model obtained will be over smoothing because the estimation used many observations. The selection of optimum bandwidth becomes very important because it will affect the accuracy of the model to the data, namely the variance and bias of the model. The method commonly used to determine the optimum bandwidth is Cross Validation (CV) which is formulated as follows [10]:

$$CV(b) = \sum_{i=1}^{n} \left(y_i - \hat{y}_{\neq i}(b)\right)^2$$

with $\hat{y}_{\neq i}(b)$ is the estimation value of y_i where the observation in i-th location is not included in the estimation process. This approach tests the model only with samples close to i-th point, not at i-th point itself. An optimal b value will be obtained on a minimum CV.

2.7 Mean Square Error (MSE)

Definition 2.1 (Mean Square Error)

Mean square error of point estimate $\hat{\theta}$ is

$$MSE(\hat{\theta}) = E\left[(\hat{\theta} - \theta)^2\right].$$

MSE can be written as the sum of the variance of the estimator and the squared bias of the estimator. If $B(\hat{\theta})$ represents the bias of the estimator $\hat{\theta}$, then it can be stated as follows [27],

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + [B(\hat{\theta})]^2. \tag{12}$$

Proof of Equation 12

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$$

$$= E[(\hat{\theta} - E(\hat{\theta}) + E(\hat{\theta}) - \theta)^2]$$

$$= E\left[\left(\hat{\theta} - E(\hat{\theta})\right)^2 + 2\left(\hat{\theta} - E(\hat{\theta})\right)\left(E(\hat{\theta}) - \theta\right) + \left(E(\hat{\theta}) - \theta\right)^2\right]$$

$$= E\left[\left(\hat{\theta} - E(\hat{\theta})\right)^2\right] + E\left[2\left(\hat{\theta} - E(\hat{\theta})\right)\left(E(\hat{\theta}) - \theta\right)\right] + E\left[\left(E(\hat{\theta}) - \theta\right)^2\right]$$

$$= E\left[\left(\hat{\theta} - E(\hat{\theta})\right)^2\right] + 2\left(E(\hat{\theta}) - \theta\right)E\left[\left(\hat{\theta} - E(\hat{\theta})\right)\right] + E\left[\left(E(\hat{\theta}) - \theta\right)^2\right]$$

$$= E\left[\left(\hat{\theta} - E(\hat{\theta})\right)^2\right] + 2\left(E(\hat{\theta}) - \theta\right)\left(E(\hat{\theta}) - E(\hat{\theta})\right) + E\left[\left(E(\hat{\theta}) - \theta\right)^2\right]$$

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + [B(\hat{\theta})]^2.$$

2.8 Akaike's Information Criterion (AIC)

To compare and evaluate the quality of the set models resulted using the two methods. Akaike's Information Criterion (AIC) will be used. AIC is the criteria for goodness of fit model by estimating the model statistically. AIC criteria are usually used when the formation of a regression model aims to obtain factors that affect the model not to make a prediction.

The size of the AIC is in line with the deviation value of the model. The smaller deviation value, the smaller error rate produced by the model so that the model obtained becomes more precise. Therefore, the best model is model with the smallest AIC. AIC values can be calculated using the following formulas [28]

$$AIC = 2p - 2\ln(\hat{L})$$

where, $\hat{L}$ is maximum value of the likelihood function and p is the number of estimated parameters.

III. RESEARCH METHOD

3.1 Research Time and Place

This research was conducted in the even semester of the academic year 2021/2022 and took place in the Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Lampung.

3.2 Research Method

In this study, we used R software version 4.1.2 to facilitate the simulation study and the Monte Carlo simulation was used to compare SAR and GWR models. The map of Indonesia also used as a spatial reference in this simulation study. Indonesia is composed of 34 provinces so that we have 34 spatial units. The Monte Carlo simulation was conducted using 1000 samples.

The scheme of the simulation in this study is as follows [29]:

1. Generate variable bivariate normal X_1 and X_2 with $n = 34$, $\mu = 0$ and $\sigma^2 = 1$.
2. Generate weights matrix ($\mathbf{W}$) using Queen contiguity.
3. Generate $\boldsymbol{\varepsilon}$ with normal distribution, $\mu = 0$ and $\sigma^2 = 1$.
4. Generate variable dependent y for conditional spatial dependency
   $\mathbf{y} = (\mathbf{I} - \rho\mathbf{W})^{-1}(\beta_0 + \beta_1\mathbf{X_1} + \beta_2\mathbf{X_2}) + (\mathbf{I} - \rho\mathbf{W})^{-1}\boldsymbol{\varepsilon}$
   with $\beta_0 = \beta_1 = \beta_2 = 1$ using $\rho = 0.7$.
5. Generate variable dependent y for conditional spatial heterogeneity
   $\mathbf{y} = (\boldsymbol{\beta_0} + \boldsymbol{\beta_1}\mathbf{X_1} + \boldsymbol{\beta_2}\mathbf{X_2}) + \boldsymbol{\varepsilon}$
   with $\boldsymbol{\beta_0} = \mathbf{1} + 0.1Lat + 0.1Long$
   $\boldsymbol{\beta_1} = \mathbf{1} + 0.2Lat + 0.2Long$

$\boldsymbol{\beta_2} = \mathbf{1} + 0.1Lat + 0.1Long.$

6. Generate variable dependent y

   $\mathbf{y} = (\mathbf{I} - \rho\mathbf{W})^{-1}(\boldsymbol{\beta_0} + \boldsymbol{\beta_1}\mathbf{X_1} + \boldsymbol{\beta_2}\mathbf{X_2}) + (\mathbf{I} - \rho\mathbf{W})^{-1}\boldsymbol{\varepsilon}$

   with $\boldsymbol{\beta_0} = \mathbf{1} + 0.1Lat + 0.1Long$

   $\boldsymbol{\beta_1} = \mathbf{1} + 0.2Lat + 0.2Long$

   $\boldsymbol{\beta_2} = \mathbf{1} + 0.1Lat + 0.1Long$

   using $\rho = 0$.

7. Repeat steps 1 to 6 1000 times.
8. Calculate the bias, MSE and AIC of each model and visualize it using plot.
9. Repeat steps 1 to 8 without steps 4 and 5 using $\rho = 0.3$, $\rho = 0.5$ and $\rho = 0.7$.
10. Evaluate/compare SAR and GWR model based on bias, MSE and AIC.

V. CONCLUSION

Based on the analytical result that have been presented in the previous chapter, it has been proven that $\widehat{\boldsymbol{\beta}}_{\boldsymbol{SAR}}$ is unbiased estimator for $\boldsymbol{\beta}$ on SAR model when data only contain spatial dependency and $\widehat{\boldsymbol{\beta}}_{\boldsymbol{GWR}}$ is unbiased estimator for $\boldsymbol{\beta}$ on GWR model when data only contain spatial heterogeneity.

Based on the results of simulation study, when $\rho = 0$, the bias, variance and MSE of $\widehat{\boldsymbol{\beta}}_{\boldsymbol{SAR}}$ are always smaller than $\widehat{\boldsymbol{\beta}}_{\boldsymbol{GWR}}$. This is because when $\rho = 0$, the data does not contain spatial dependency so the GWR method would be better in modeling data that only contain spatial heterogeneity. Nevertheless, the bias, variance and MSE of $\widehat{\boldsymbol{\beta}}_{\boldsymbol{SAR}}$ are always more stable than $\widehat{\boldsymbol{\beta}}_{\boldsymbol{GWR}}$ which gets greater when ρ also gets greater. However, when $\rho \neq 0$, the bias, variance and MSE of $\widehat{\boldsymbol{\beta}}_{\boldsymbol{GWR}}$ are always greater than $\widehat{\boldsymbol{\beta}}_{\boldsymbol{SAR}}$ although the data also contain spatial heterogeneity. Then, based on the model evaluation using AIC of each model, SAR model also have smaller AIC than GWR when the data contains spatial dependency and heterogeneity. Therefore, it can be concluded that the existence of spatial dependency on the data greatly affects the GWR method while the presence of spatial heterogeneity does not significantly affect the SAR method, both on the parameter estimator and the goodness of fit. Thus, to modeling the data that contains both spatial aspects (spatial dependency and heterogeneity), the SAR method will be better than the GWR method.

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